Inducing Peer Pressure to Promote Cooperation (Supporting Information)

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1 Model of Externalities

Consider a set of agents, N, in a social network $S = (N, E)$, where $E \subseteq N \times N$. Let $Nbr\,(i) = \{j : (i, j) \in E\}$ be the set of neighbors of the actor i. We assume that all nodes have a degree less than some constant K. Each agent $i \in N$ takes an action $x_i \in \mathbb{R}_+$ (e.g. corresponding to units of electricity consumed), and let $\mathbf{x} \in \mathbb{R}^{|N|}$ be an action profile of all agents. Each agent i experiences *raw* utility from its action/consumption defined by the function $u_i : \mathbb{R}_+ \to \mathbb{R}$. We assume u_i is a twice differentiable and strictly concave raw utility function with a finite maximum and the marginal utility u_i' approaches infinity as the action approaches zero.[1]

We present two models of externalities. The first is the standard model of externalities in which each actor's utility depends upon the actor's action and the action taken by the other actors. In the second model of externalities, the actors are able to exert peer pressure on their peers and each actor's utility also depends upon the pressure exerted by the actor on her peers and the pressure felt by the actor from her peers.

1.1 Standard Model of Externalities

In the standard treatment of externalities, it is assumed that the utility of actor i depends both on the raw utility of its own action as well as the externalities experienced due to the actions of others. The latter is captured by a function $v_i : \mathbb{R}_+ \to \mathbb{R}$, which is strictly convex and increasing, and captures the externality experienced by i due to the aggregate action of other actors in the population.[2] Therefore, the total utility of actor i, given its own action x_i and the action, $\mathbf{x}_{-i}$ of other agents $N \setminus \{i\}$ is as follows:

[1]This is a very natural case when u_i is the difference of a concave upper bounded utility and linear cost such as when the raw utility function takes the form $u_i(x_i) = f_i(x_i) - \lambda x_i$, where λ is the marginal cost of the action, and $f_i(.)$ is continuous, concave, strictly increasing, and $f_i'(0) > \lambda$ and $\exists x_i^*$ such that $\forall x > x_i^*$ we have $f_i'(x) < \lambda$. In this case, at equilibrium action x_i^*, we have $f_i'(x_i^*) = \lambda$, that is the marginal benefit from action equals the marginal cost. An example of such a function is $f_i(x) = x^a$ with $a \in (0, 1)$.

[2]For simplicity, we assume externality is negative, but the analysis should not fundamentally change with positive externalities, negative utility of action and concave externality function.

$$U_i(x_i, \mathbf{x}_{-i}) := u_i(x_i) - v_i\left(\sum_{j \neq i} x_j\right)$$

Social surplus is defined as the sum of utilities achieved by all agents in both the models.

$$S(\mathbf{x}) := \sum_{i \in N} U_i(x_i, \mathbf{x}_{-i}) \tag{1}$$

1.1.1 Equilibrium in the Standard Externalities Model

The equilibrium in the standard externalities model is well understood. There is a unique equilibrium, in which each agent takes the action that maximizes the raw utility of her action irrespective of the actions taken by other agents. Let $\mathbf{x}^*$ denote the action profile at equilibrium, and $\mathbf{x}^\circ$ denote the action profile that maximizes social surplus. In the standard model of externalities, we know that at the equilibrium, the agents take action that is higher than the socially optimal action, that is $\mathbf{x}^* > \mathbf{x}^\circ$ [1] and therefore the social surplus at the equilibrium is sub-optimal.

1.2 Externalities with Peer Pressure

Consider, now, that actors have the ability to exert peer pressure on their peers in the social network [2]. We denote the peer-pressure profile by the matrix $\mathbf{p} \in \mathbb{R}^{N \times N}$, where the element p_{ij} is the peer-pressure exerted by the agent i on her peer j. If i and j are not peers in the social network, then $p_{ij} = 0$. The ith column of $\mathbf{p}^T$ (transpose of $\mathbf{p}$), $\mathbf{p}_{\uparrow i}$, is the vector of peer-pressures exerted by i over her peers. Similarly, the ith column of $\mathbf{p}$, $\mathbf{p}_{\downarrow i}$, is the vector of peer-pressures exerted over i by her peers.

The utility of an actor then takes the following extended form:

$$U_i(x_i, \mathbf{x}_{-i}, \mathbf{p}) = u_i(x_i) - v_i\left(\sum_{j \neq i} x_j\right) - \left(\sum_{j \in Nbr(i)} p_{ji}\right)(x_i - x_i^\circ) - \left(\sum_{j \in Nbr(i)} p_{ij}\right)c \tag{2}$$

Thus, in addition to the raw utility of action and the externality, actor i also experiences potential disutility that is bilinear in the total pressure from the peers and i's own action. An individual's action and the peer pressure on the individual enter as strategic substitutes in the individual's utility. The higher i's action is, the more salient the effect of the pressure becomes (similar model with binary actions has also been studied by authors in [2, pg. 67]). Agent i also incurs a cost $c.p_{ij}$ should it wish to exert pressure on neighb1or j, where c is the marginal cost of exerting such pressure.

In the vector form, the utility takes the form:

$$U_i(x_i, \mathbf{x}_{-i}, \mathbf{p}) = u_i(x_i) - v_i\left(\mathbf{1}.\mathbf{x}_{-i}\right) - \left(\mathbf{1}.\mathbf{p}_{\downarrow i}\right)\left(x_i - x_i^{\circ}\right) - \left(\mathbf{1}.\mathbf{p}_{\uparrow i}\right)c \tag{3}$$

{We will denote $\mathbf{1}$ as a vector of ones with matching dimensions when taking the dot product.}

With peer-pressure, the game is modelled as a two-stage game. In the first stage, actors choose the amount of peer pressure they wish to exert on their neighbors. In the second stage, actors observe pressure on themselves and then choose their action. We assume that the raw marginal utility of any agent is convex in her action.[3]

The scenario we are interested in is when the externality functions rise much slower than the raw utility function and any one agent's marginal change in action has very small effect on the marginal externality of other agents. In this case, the large externality is due to higher aggregate action of all agents and in the equilibrium the peer-pressure is sufficiently distributed across all agents. We also restrict to the cases when the marginal cost of exerting peer-pressure is neither too high such that no one exerts any peer-pressure on anyone, nor too low such that there is too much pressure on everyone. This is often the case with problems of externalities such as pollution. The change in externality felt by any one agent due to the change in one other agent's pollution level is significantly small compared to the gain the utility of the agent creating the pollution. The total change in the externality felt by all the agents may be very high for a large population of agents. Similarly, if a large number of agents change their pollution level simultaneously, then the change in the externality felt by any one agent may be much larger than the change in the utility of any one agent changing her pollution level. More formally, we make the following assumptions.

1. For at least one agent i and one of her peers j, the marginal cost of exerting pressure is lower than the ratio of the marginal externality of j to the curvature of the raw utility of i when the action profile is $\mathbf{x}^*$, i.e., c $< \left|\frac{v_j'\left(\mathbf{1}.\mathbf{x}_{-j}^*\right)}{u_i''\left(x_i^*\right)}\right|$. In the absence of this condition, the cost of exerting peer-pressure is too high and no one will exert peer-pressure on anyone. Since, we observe peer-pressure in the real world, this condition is naturally satisfied.

2. The marginal cost of exerting peer-pressure is higher than the ratio of the marginal externality of any agent, i, to the curvature of the raw utility of any of her peers, j, when the action profile is socially optimal $\mathbf{x}^{\circ}$, i.e., $c > \left|\frac{v_j'\left(\mathbf{1}.\mathbf{x}_{-j}^{\circ}\right)}{u_i''\left(x_i^{\circ}\right)}\right|$ for all peers i, j. This

[3]This assumption is very natural and satisfied by almost all natural utility functions such as the Cobb-Douglas utility function.

condition says that the cost of exerting pressure high enough that no agent experiences enough pressure to reduce his action to the socially optimal level. In the absence of this condition, all actions in the society will be optimal. Since, there definitely are suboptimal actions in the society, therefore, this condition is naturally satisfied.

3. The marginal cost of exerting pressure is $\frac{1}{2K}$ times lower than the reciprocal of the absolute semi-elasticity of the marginal raw utility of any agent i at the socially optimal action profile, $\mathbf{x}^\circ$, i.e., $c < \frac{1}{2K\left|\frac{\partial \log u_i'\left(x_i^\circ\right)}{\partial x_i^\circ}\right|} = \frac{1}{2K}\left|\frac{u_i'\left(x_i^\circ\right)}{u_i''\left(x_i^\circ\right)}\right| = \frac{1}{2K}\left|\frac{\sum_{j\neq i} v_j'\left(\mathbf{1}.\mathbf{x}_{-j}^\circ\right)}{u_i''\left(x_i^\circ\right)}\right|$.

For convenience, we define some notation:

- Let $p_{\downarrow i} := \sum_{j\in Nbr(i)} p_{ji} = \mathbf{1}.\mathbf{p}_{\downarrow i}$ denote the total pressure exerted over i by all his neighbors.
- Let $p_{i\uparrow} := \sum_{j\in Nbr(i)} p_{ij} = \mathbf{1}.\mathbf{p}_{\uparrow i}$ denote the total pressure exerted by i towards his neighbors.

We study the equilibrium of this game in the next sections.

2 Equilibrium in the Externalities Model with Peer Pressue

We now study the sub-game perfect equilibrium in the externalities model with peer-pressure. In the sub-game perfect equilibrium, we observe that the equilibrium action profile in the game with peer-pressure is lower than the action-profile in the game without peer-pressure. The presence of peer-pressure causes a lower action in the game. However, the peer-pressure in the equilibrium is not sufficient to bring the action down to the socially optimal level.

2.1 Existence of the Subgame Perfect Equilibria

We first show how the peer pressure profile in the first stage effects the equilibrium in the second stage game. In the second stage of the game for the given pressure profile $\mathbf{p}$, each actor i chooses her action x_i that maximizes her utility $U_i(x_i, \mathbf{x}_{-i}, \mathbf{p})$ given the peer-pressure exerted on her. Note that an actor's marginal utility in her own action in the subgame starting at the second stage is independent of the actions of the other actors in the second stage. Only the first three terms in the utility function depend upon the action profile in the second stage and only the first and the third term depends upon an actor's own action x_i. Therefore, in essence, the sub-game starting at the second stage is similar to the game in the standard externalities model. In the second stage, actors observe pressure on themselves and then choose their action as a response to the observed pressure. Given pressure $\mathbf{p}$, we denote $x_i^*(\mathbf{p})$ or with some abuse of notation, $x_i^*(p_{\downarrow i})$ as the optimal response to the pressure for agent i, which is also his equilibrium action in the second stage game. Since the marginal raw utility of action for all actors approaches infinity as action approaches zero, therefore this action for all actors is greater than zero. The optimal response is unique and the marginal raw utility for actor i at the optimal response is equal to the total pressure exerted on i, i.e. $u_i'(x_i^*(p_{\downarrow i})) = p_{\downarrow i}$. The marginal response by any agent i to pressure p_{ji} from any of her peers j is the reciprocal of the curvature of the raw utility function at the optimal response, i.e.- $\frac{\partial x_i^*(p_{\downarrow i})}{\partial p_{ji}} = \frac{1}{u''(x_i^*(p_{\downarrow i}))}$ [4]. Since, the raw utility of action is concave, the optimal response in the second stage is decreasing in the felt peer-pressure, i.e.- $\frac{\partial x_i^*(p_{\downarrow i})}{\partial p_{\downarrow i}} = \frac{1}{u_i''(x_i^*(p_{\downarrow i}))} < 0$. Also, since the raw marginal utility of action is convex in the agent's action, therefore, the optimal response in the second stage is convex in the felt peer-pressure, $\frac{\partial^2 x_i^*(p_{\downarrow i})}{\partial p_{\downarrow i}^2} > 0$.

In the first stage of the game, the threat of higher action in the second stage, is an incentive for the actors to exert pressure on their peers. Given that the equilibrium action is taken in the second stage of the game, the actors in the first stage of the game choose peer-pressure on their neighbors that maximixes their utility function

$$U_i(\mathbf{p}) = u_i(x_i^*(p_{\downarrow i})) - v_i\left(\sum_{j\neq i} x_j^*(p_{\downarrow j})\right) - p_{\downarrow i}x_i^*(p_{\downarrow i}) - p_{\uparrow i}c \tag{4}$$

2.1.1 Peer Pressures are strategic substitutes

Proposition 1. *The peer-pressure that any actor i exerts on any of her peers j is strict strategic substitute of any peer-pressure not exerted on i. Therefore, if any agent increases peer-pressure on one of her neighbors other than i, then the best-reponse peer-pressure exerted by i on all other agents decreases.*

Proof. For any actor i and any two of her peers j, k,
$\frac{\partial^2 U_i}{\partial p_{ij}\partial p_{ik}} = -v_i''\left(\sum_{m\neq i} x_m^*(p_{\downarrow m})\right)\frac{\partial x_j^*(p_{\downarrow j})}{\partial p_{ij}}\frac{\partial x_k^*(p_{\downarrow k})}{\partial p_{ik}} < 0$, since the externality is concave in the action and the best response action of the actors is decreasing in the felt peer-pressure. Therefore, the peer-pressures p_{ij} and p_{ik} are strict strategic substitutes of each other.

For any actor i and k with a common peer j,
$\frac{\partial^2 U_i}{\partial p_{ij}\partial p_{kj}} = -v_i''\left(\sum_{m\neq i} x_m^*(p_{\downarrow m})\right)\frac{\partial x_j^*(p_{\downarrow j})}{\partial p_{ij}}\frac{\partial x_j^*(p_{\downarrow j})}{\partial p_{kj}} - v_i'\left(\sum_{m\neq i} x_m^*(p_{\downarrow m})\right)\frac{\partial^2 x_j^*(p_{\downarrow j})}{\partial p_{ij}\partial p_{kj}} < 0$, since the externality is concave in the action and the best response action of the actors is decreasing and convex in the felt peer-pressure. Therefore, the peer-pressures p_{ij} and p_{kj} are strict

[4] By using the inverse derivative law: : $(f^{-1})' = \frac{1}{f'(f^{-1})}$

strategic substitutes of each other.

For any distinct pair of neighboring actors i, j and k, l, $\frac{\partial^2 U_i}{\partial p_{ij} \partial p_{kl}} = -v_i'' \left(\sum_{m \neq i} x_m^* \left(p_{\downarrow m} \right) \right) \frac{\partial x_j^* \left(p_{\downarrow j} \right)}{\partial p_{ij}} \frac{\partial x_l^* \left(p_{\downarrow l} \right)}{\partial p_{kl}} < 0$, since the externality is concave in the action and the best response action of the actors is decreasing in the felt peer-pressure. Therefore, the peer-pressures p_{ij} and p_{kl} are strict strategic substitutes of each other.

□

2.1.2 Eliminating Some Strictly Dominated Strategies

Since, all peer pressures are positive and the optimal response of the agents decrease with the felt pressure, the optimal response of any agent in the second stage game is always below the equilibrium action in the standard externalities model, x_i^*. Therefore, all actions $x_i > x_i^{max} = x_i^*$ and are strictly dominated. Let us denote $p_{ji}^{max} = \arg\max_{p_{ji} \in \mathbb{R}_+} U_j \left(x_i^* \left(p_{ji} \right), \mathbf{x}_{-i}^*, p_{ji}, \mathbf{0} \right)$, the pressure that j puts on i that maximizes j's utility given she does not put any peer-pressure on any other peer and no one else puts any peer-pressure on anyone else. Following Proposition 1, any peer pressure strategy $\mathbf{p}_{\uparrow j}$ in which $p_{ji} > p_{ji}^{max}$ is strictly dominated. Since the maximum pressure on i from any of her peers is finite, therefore the maximum total pressure $p_{\downarrow i}^{max}$ on agent i is finite and since i's marginal raw utility of action approaches infinity as her action approaches zero, therefore all actions $x_i < x_i^* \left(p_{\downarrow i}^{max} \right) = x_i^{min}$ in the second stage game are strictly dominated.

Therefore, for any agent i, there exists critical actions, x_i^{min} and x_i^{max}, such that all actions not belonging to $[x_i^{min}, x_i^{max}]$ are iteratively strictly dominated. Similarly, there exists a maximum peer pressure, p_{ij}^{max} such that for any pair of peers i, j, all $p_{ji} > p_{ij}^{max}$ are strictly dominated. There are more iterated strictly dominated strategies but we will restrict our attention to these.

Therefore, we can restrict the set of actions for any actor i to $X_i = [x_i^{min}, x_i^{max}]$ and peer-pressure strategy set for any actor i to $\mathbf{P}_i = \prod_{j \in Nbr(i)} [0, p_{ij}^{max}]$ for all $\mathbf{p}_{-i}$(peer pressure profile of all agents other than i) without any loss of generality. The set of peer-pressure strategy profiles $\mathbf{P} = \prod_{i \in N} \mathbf{P}_i$ is also compact and convex.

2.1.3 Existence of Subgame Perfect Equilibrium

Proposition 2. *The set of subgame perfect equilibria in the two stage game is non-empty.*

Proof. We first show that for all $i \in N$ and for all $\mathbf{p}_{-i}$, $v_i \left(\sum_{j \neq i} x_j^* \left(\mathbf{p}_{-i}, \mathbf{p}_{i\uparrow} \right) \right)$ is strictly convex in $\mathbf{p}_{i\uparrow}$. To show this, we consider two arbitrary peer-pressure vectors $\mathbf{p}_i'$ and $\mathbf{p}_i''$.

$$v_i\left(\sum_{j\neq i} x_j^*\left(\mathbf{p}_{-i}, \lambda \mathbf{p}'_{i\uparrow} + (1-\lambda)\,\mathbf{p}''_{i\uparrow}\right)\right)$$

$$= v_i\left(\sum_{j\in N\setminus(Nbr(i)\cup\{i\})} x_j^*\left(\mathbf{p}_{-i}, \lambda \mathbf{p}'_{i\uparrow} + (1-\lambda)\,\mathbf{p}''_{i\uparrow}\right) + \sum_{j\in Nbr(i)} x_j^*\left(\mathbf{p}_{-i}, \lambda \mathbf{p}'_{i\uparrow} + (1-\lambda)\,\mathbf{p}''_{i\uparrow}\right)\right)$$

$$< v_i\left(\sum_{j\in N\setminus(Nbr(i)\cup\{i\})} x_j^*\left(\mathbf{p}_{-i}, \lambda \mathbf{p}'_{i\uparrow} + (1-\lambda)\,\mathbf{p}''_{i\uparrow}\right) + \sum_{j\in Nbr(i)} \lambda x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}'_{i\uparrow}\right) + (1-\lambda)\,x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}''_{i\uparrow}\right)\right)^5$$

$$= v_i\left(\sum_{j\in N\setminus(Nbr(i)\cup\{i\})} x_j^*\left(\mathbf{p}_{-i}\right) + \sum_{j\in Nbr(i)} \lambda x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}'_{i\uparrow}\right) + (1-\lambda)\,x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}''_{i\uparrow}\right)\right)^6$$

$$< \lambda v_i\left(\sum_{j\in N\setminus(Nbr(i)\cup\{i\})} x_j^*\left(\mathbf{p}_{-i}\right) + \sum_{j\in Nbr(i)} x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}'_{i\uparrow}\right)\right)$$
$$+ (1-\lambda)\,v_i\left(\sum_{j\in N\setminus(Nbr(i)\cup\{i\})} x_j^*\left(\mathbf{p}_{-i}\right) + \sum_{j\in Nbr(i)} x_j^*\left(\mathbf{p}_{-i}, \mathbf{p}''_{i\uparrow}\right)\right)^7$$

.

Therefore the utility, U_i, of actor i is strictly concave in $\mathbf{p}_{\uparrow i}$ for all $\mathbf{p}_{-i}$. Also, since v_i approaches $-\infty$ as its paramater approaches ∞, therefore there is a unique and finite $\mathbf{p}_{\uparrow i}$ for each $\mathbf{p}_{-i}$ that maximizes i's utility.

Since, the peer-pressure strategy set, $\mathbf{P}_i$ for any actor i is compact and convex for any $\mathbf{p}_{-i}$ and the utility function of i is continuous and strictly concave, therefore by the theorem of the maximum [3], the best-response peer-pressure strategy, $\mathbf{p}^*_{\uparrow i}(\mathbf{p})$, of actor i is continuous in $\mathbf{p}$ and the best-response peer-pressure function $\mathbf{p}^* : \mathbf{P} \to \mathbf{P}$ is a continuous function from a compact and convex set to itself. Therefore, by Brower's fixed point theorem, the best-response peer-pressure function $\mathbf{p}^*$ has a fixed point. Therefore, the set of equilibria is non-empty.

□

[5]Since, for all $j \in Nbr(i)$, $x_j^*\left(p_{\downarrow j}\right)$ is strictly convex in $p_{\downarrow j}$ and as a consequence in p_{ij}

[6]Since, for all $j \notin Nbr(i)$, $p_{\downarrow j}$ is independent of $p_{\uparrow i}$, therefore $x_j^*(\mathbf{p}) = x_j^*\left(p_{\downarrow j}\right)$ is independent of $\mathbf{p}_{\uparrow i}$

[7]Since v_i is strictly convex

2.2 Who puts pressure on whom?

We make two observations for the equilibrium pressure profile $\mathbf{p}^*$ that follow from applying the Karush-Kuhn-Tucker(KKT) conditions [4].

Proposition 3. *The set of peers an actor puts pressure on are the ones who have the smallest curvature of the raw utility function at the equilibrium action, i.e.- $p_{ji} > 0$ for $j \in N$, $i \in Nbr\,(i)$ only if $u_i''\left(x_i^*\left(p_{\downarrow i}^*\right)\right) \leq u_k''\left(x_k^*\left(p_{\downarrow k}^*\right)\right)$ for all $k \in Nbr\,(i)$.*

Proof. For the actor j, if $p_{ji}^* > 0$, then by applying KKT condition to the equilibrium, we have:

$$c = -v_j'\left(\sum_{k \neq j} x_k^*(\mathbf{p}^*)\right)\frac{1}{u_i''\left(x_i^*(p_{\downarrow i}^*)\right)}$$

Therefore for any peer i on whom j exerts presure, the curvature of i's raw utility function $\left|u_i''\left(x_i^*(p_{\downarrow i}^*)\right)\right| = \frac{c}{v_j'\left(\sum_{k \neq j} x_k^*(\mathbf{p}^*)\right)}$. and if $p_{jk}^* = 0$ for some $k \in Nbr\,(j)$.

$$c > -v_j'\left(\sum_{k \neq j} x_k^*(\mathbf{p}^*)\right)\frac{1}{u_k''\left(x_k^*(p_{\downarrow k}^*)\right)}$$

Therefore for any peer k on whom j does not exert presure, the curvature of k's raw utility function $\left|u_i''\left(x_i^*(p_{\downarrow i}^*)\right)\right| > \frac{c}{v_j'\left(\sum_{k \neq j} x_k^*(\mathbf{p}^*)\right)}$. Therefore the proposition holds.

□

In words, agent j will exert pressure on every neighbor for which the marginal cost of exerting pressure c equals the marginal reduction in externalities experienced from that neighbor. Moreover, note that all these neighbors will have identical marginal reduction in externality. On the other hand, j will not exert any pressure on the other neighbors, for which the marginal reduction in externality is lower than c.

Proposition 4. *An actor feels pressure from the peers who have the highest marginal externality at the equilibrium action, i.e.- $p_{ji} > 0$ for $i \in N$, $j \in Nbr\,(i)$ only if $v_j'\left(\mathbf{1}.\mathbf{x}^*\left(\mathbf{p}^*\right) - x_j^*\left(p_{\downarrow j}^*\right)\right) \geq v_k'\left(\mathbf{1}.\mathbf{x}^*\left(\mathbf{p}^*\right) - x_k^*\left(p_{\downarrow k}^*\right)\right)$ for all $k \in Nbr\,(i)$.*

Proof. For the actor i, if $p_{ji}^* > 0$, then

$$c\left|u_i''\left(x_i^*(p_{\downarrow i}^*)\right)\right| = v_j'\left(\sum_{l \neq j} x_l^*(\mathbf{p}^*)\right)$$

and for all $k \in Nbr(i)$ if $p^*_{ki} = 0$.

$$c\left|u''_i\left(x^*_i(p^*_{\downarrow i})\right)\right| > v'_k\left(\sum_{l\neq k} x^*_l(\mathbf{p}^*)\right)$$

Therefore the proposition holds.

$\square$

The above observations provide a system of equations that can be solved to find an equilibrium pressure profile $\mathbf{p}^*$ and the resulting action profile $\mathbf{x}^*$. We note that the equilibrium pressure profile need not be unique. However, as we see in the next subsection, the total pressure on each agent and consequently the action profile is same in all equilibria.

2.3 Uniqueness of action profile in all subgame perfect equilibria

Assumption A1: the absolute elasticity of the marginal externality felt by any agent $j \in N$ is lower than the absolute elasticity of the curvature of the raw utility of any other agent $i \in N$ with respect to the agent i's action, i.e.- $\frac{\partial \log v'_j\left(x_i + \mathbf{1}.\mathbf{x}_{-\{i,j\}}\right)}{\partial x_i} < \left|\frac{\partial \log\left|u''_i(x_i)\right|}{\partial x_i}\right|$ for all action profiles $\mathbf{x} \in \prod_{k\in N}[x^{min}_k, x^{max}_k]$.

This assumption suggests that the marginal externalities grow slower than the decay of curvature of the raw utility function. We will see that this condition guarantees that the peer-pressure on any agent and consequently her own action does not vary across different equilibria. We first give a result that will help understand the implication of assumption A1.

Proposition 5. *Define $h(x,y) = -af(x) + bg(y)$ and where f and g are continuous, differentiable and positive functions where f is strictly decreasing and g is strictly increasing defined over closed intervals. Assume $\frac{\partial \log g(y)}{\partial y} < -\frac{\partial \log f(x)}{\partial x}$. If $h(x_1, y_1) = h(x_2, y_2) = 0$, with $y_2 > y_1$ and $x_2 < x_1$. Then $x_1 - x_2 < y_2 - y_1$.*

Proof. Let l be the path from (x_1, y_1) to (x_2, y_2) along the indifference curve of h, i.e.- $h(x,y) = 0$ for all (x,y) along l.

$$h(x_2, y_2) - h(x_2, y_2) = \int_{x_1,y_1}^{x_2,y_2} \nabla h(x,y).dl = 0$$

$$\implies \int_{x_1,y_1}^{x_2,y_2} \left(-af'(x)\,dx + bg'(y)\,dy\right).dl = 0$$

$$\implies \int_{x_1,y_1}^{x_2,y_2} \left(-\frac{af'(x)}{af(x)}dx + \frac{bg'(y)}{bg(y)}dy\right).dl = \int_{x_1,y_1}^{x_2,y_2} \left(-\frac{f'(x)}{f(x)}dx + \frac{g'(y)}{g(y)}dy\right).dl = 0$$

{Since $af(x) = bg(y)$ for all (x, y) along l}

$$\implies \int_{x_1,y_1}^{x_2,y_2} \left(-\frac{\partial \log f(x)}{\partial x} dx + \frac{\partial \log g(y)}{\partial y} dy\right).dl = 0$$

$$\implies \int_{x_1,y_1}^{x_2,y_2} -dx.dl < \int_{x_1,y_1}^{x_2,y_2} dy.dl$$

{Since -$\frac{\partial \log f(x)}{\partial x} > \frac{\partial \log g(y)}{\partial y} > 0$ for all (x, y) along l}

$$\implies x_1 - x_2 < y_2 - y_1$$

□

Using the above result, we show the uniqueness of action profile in the two-stage game with peer pressure.

Proposition 6. *The action profile in the second stage of the game is same in all subgame perfect equilibria of the two-stage game under assumption A1.*

Proof. We will prove this by contradiction. Assume, there are two subgame perfect equilibria $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$, $(\breve{\mathbf{p}}, \breve{\mathbf{x}})$ with two different action profiles $\hat{\mathbf{x}}$ and $\breve{\mathbf{x}}$ respectively. Without loss of generality, assume that $\mathbf{1}'.\hat{\mathbf{x}} \geq \mathbf{1}'.\breve{\mathbf{x}}$. Let $i^* = \arg\max_{i \in N} \hat{x}_i - \breve{x}_i$. Clearly, $\hat{x}_{i^*} - \breve{x}_{i^*} > 0$. This implies that $\breve{p}_{\downarrow i^*} > \hat{p}_{\downarrow i^*} \geq 0$. Pick a peer, $j^* \in Nbr(i^*)$ such that $\breve{p}_{j^* i^*} > 0$. We know that such a neighbor exists because $\breve{p}_{\downarrow i^*} > 0$.

Therefore by KKT condition,

$$cu''_{i^*}(\breve{x}_{i^*}) + v'_{j^*}\left(\sum_{k \neq j^*} \breve{x}_k\right) = 0$$

If

$$cu''_{i^*}(\hat{x}_{i^*}) + v'_{j^*}\left(\sum_{k \neq j^*} \hat{x}_k\right) = 0$$

then by the previous proposition

$$(\mathbf{1}'.\breve{\mathbf{x}} - \breve{x}_{j^*}) - (\mathbf{1}'.\hat{\mathbf{x}} - \hat{x}_{j^*}) > \hat{x}_{i^*} - \breve{x}_{i^*}$$

If

$$cu''_{i^*}(\hat{x}_{i^*}) + v'_{j^*}\left(\sum_{k \neq j^*} \hat{x}_k\right) < 0$$

then define $x^{\cdot}_{i^*}$ to be the action of i such that

$$cu''_{i^*}\left(x^{\cdot}_{i^*}\right) + v'_{j^*}\left(\sum_{k\neq j^*}\hat{x}_k\right) = 0$$

Since the raw marginal utility of action is convex, therefore $\hat{x}_{i^*} < x^{\cdot}_{i^*}$ and therefore using the previous proposition

$$\left(\mathbf{1}'.\breve{\mathbf{x}} - \breve{x}_{j^*}\right) - \left(\mathbf{1}'.\hat{\mathbf{x}} - \hat{x}_{j^*}\right) > x^{\cdot}_{i^*} - \breve{x}_{i^*} > \hat{x}_{i^*} - \breve{x}_{i^*}$$

This implies that

$$\hat{x}_{j^*} - \breve{x}_{j^*} > \hat{x}_{i^*} - \breve{x}_{i^*} + \mathbf{1}'.\hat{\mathbf{x}} - \mathbf{1}'.\breve{\mathbf{x}} > \hat{x}_{i^*} - \breve{x}_{i^*} = \max_{i\in N}\hat{x}_i - \breve{x}_i \geq \hat{x}_{j^*} - \breve{x}_{j^*}$$

which is a contradiction. Therefore, there cannot be two subgame perfect equilibria with different action profiles.

□

2.4 Comparing the Equilibrium Action in the Two Games

Theorem 1. *The action profile in any subgame perfect equilibrium in the two-stage game with peer-pressure is strictly lower than the equilibrium action profile in the game without peer-pressure.*

Proof. The action profile in the equilibrium in the game with peer-pressure is same as the action profile in the equilibrium of the second stage in the two-stage game when the peer-pressure profile in the first stage is 0. Since the equilibrium action profile in the second stage decreases with any increase in the peer-pressure profile in the first stage and the peer-pressure profile in the subgame perfect equilibrium of the two-stage game is positive, therefore the result holds.

□

3 Pigouvian Mechanism: Direct Rewards to Reduce Action to the Socially Optimal Level

In the standard Pigouvian mechanism, agents are rewarded for reduced action. The reward given to any agent $i \in N$ for her action x_i is $r_i\left(x_i\right) = u'_i\left(x^\circ_i\right)\left(x^*_i - x_i\right)$. The utility function of agent i under the Pigouvian mechanism is

$$U_i(x_i, x_{-i}, \mathbf{p}) = u_i(x_i) - v_i\left(\sum_{j \neq i} x_j\right) - \left(\sum_{j \in Nbr(i)} p_{ji}\right) x_i - \left(\sum_{j \in Nbr(i)} p_{ij}\right) c + r_i\left(x_i\right). \quad (5)$$

Proposition 7. *There is no peer-pressure on any agent in equilibrium under Pigouvian mechanism.*

Proof. When there is no peer-pressure on any agent, the equilibrium action profile is $\mathbf{x}^\circ$. For any pair of peers i,j, $c > -\frac{v_j'(\mathbf{x}^\circ)}{u_i''(x_i)}$, therefore by KKT conditions the best response peer-pressure for agent j, $p_{ij} = 0$. Therefore there is no peer-pressure on any agent in the equilibrium under Pigouvian mechanism.

□

4 Social Mechanisms: Rewards to Increase Peer Pressure to the Socially Optimal Level

Social mechanisms encourage individuals to exert peer pressure and thus indirectly reduce the action level. These mechanisms work by rewarding individuals for their peers' low action, in effect subsidizing the cost of peer pressure they incur. The reward is given to agent i as a result of her peer, agent j's action x_j. Formally, $r_{ji} : \mathbb{R}_+ \to \mathbb{R}$ is strictly decreasing in x_j (the more j consumes, the less reward i gets), and $r_{ji}\left(x_j\right)$ is the reward given to the actor i for the action x_j by a neighboring actor j.

This reward can be incorporated into the game by the following expanded utility function:

$$U_i(\mathbf{x}, \mathbf{p}) = u_i(x_i) - v_i\left(\sum_{j \neq i} x_j\right) - x_i \sum_{j \in Nbr(i)} p_{ji} - c \sum_{j \in Nbr(i)} p_{ij} + \sum_{j \in Nbr(i)} r_{ji}(x_j)$$

At optimal action x°, $\forall i \in N$, we have $u_i'(x_i^\circ) = \sum_{j \neq i} v_j'(\sum_{k \in N} x_k^\circ)$

We would like to come up with a reward function which has the following properties:

1. The reward must be simple. We consider reward functions with constant marginal reward (i.e. affine reward functions).

2. A subgame perfect equilibrium of the two-stage game should exist.

3. Equilibrium action should be optimal.

4. Each peer gets rewarded for an agent's reduced action.

5. Budget for rewards should be minimized over the set of reward functions that satisfy the above conditions.

Theorem 2. *The following reward function satisfies conditions (1-5):*

$$r_{ji}(x_j) = (\alpha_j + \beta_i)\left(x_j - x_j^*\right) \text{ where } \alpha_j = cu_j''(x_j^\circ) \text{ and } \beta_i = v_i'\left(\sum_{k\neq i} x_k^\circ\right)$$

Proof. In the first stage, the actors choose the pressure such that action profile in the second stage maximizes their reward minus the externality on them minus the cost they incur to exert pressure. The optimum amount of pressure is such that the action profile in the second stage is socially optimal. The threat of over action reducing the reward for the actors in the second stage sustains the optimum pressure exerted by the actors. Thus, at the optimum pressure profile $\mathbf{p}^\circ$, we should have in the second stage:

$$\left(-v_i'\left(\sum_{k\neq i} x_k^*(\mathbf{p})\right)\frac{\partial x_j^*(\mathbf{p})}{\partial p_{ij}} - c + r_{ji}'\left(x_j^*(\mathbf{p})\right)\frac{\partial x_j^*(\mathbf{p})}{\partial p_{ij}}\right)\Bigg|_{\mathbf{p}^\circ} = 0$$

which gives:

$$\left(\left(r_{ji}'(x_j^*(\mathbf{p})) - v_i'\left(\sum_{k\neq i} x_k^*(\mathbf{p})\right)\right)\frac{\partial x_j^*(\mathbf{p})}{\partial p_{ij}}\right)\Bigg|_{\mathbf{p}^\circ} = c$$

Therefore:[8]

$$r_{ji}'(x_j^\circ) = cu_j''(x_j^\circ) + v_i'\left(\sum_{k\neq i} x_k^\circ\right)$$

This characterizes the reward function such that the pressure profile in the first stage is optimum. This is not a full characterization of the reward function as it only suggests constraints on the reward function such that the optimal pressure is chosen in the first stage. There are infinitely many reward functions that satisfy these constraints. In short the constraint suggests that at action levels below the optimal, the marginal reward of the neighbors additional action is higher than the sum of the marginal reduction in cost by reducing pressure on the neighbor and the marginal externality introduced by the neighbors action. Thus, it is advantageous for the actors to reduce pressure on their neighbors. On the other hand, at action levels above optimum, the marginal reward for the neighbors additional action is lower than the sum of the marginal reduction in cost by reducing pressure on

[8]The inverse derivative: $\frac{\partial x_j^*(\mathbf{p})}{\partial p_{ij}} = \frac{1}{u''(x_j^*(\mathbf{p}))}$ using the inverse derivative law: $\left(f^{-1}\right)' = \frac{1}{f'(f^{-1})}$

the neighbor and the marginal externality introduced by the neighbors action. Thus, it is advantageous for the actors to increase pressure on their neighbors. To fully characterize the reward function, we use conditions 1 and 2.

The following reward function satisfies the above condition:

$$r_{ji}(x_j) = -(\alpha_j + \beta_i)\left(x_j^* - x_j\right) \text{ where } \alpha_j = cu_j''(x_j^\circ) \text{ and } \beta_i = v_i'\left(\sum_{k \neq i} x_k^\circ\right)$$

□

The reward has a component that depends upon the consumer and a component that depends upon the neighbor.

4.1 Comparing the budget for rewards for the social and the Pigouvian mechanisms

Theorem 3. *The budget for the rewards in the Pigouvian Mechanism is at least twice the budget for the rewards in the social mechanism.*

Proof. The budget for the total reward distributed to the peers of agent j at the optimal action profile is

$$B_j^s = \sum_{i \in Nbr(j)} r_{ji}(x_j^\circ) = \sum_{i \in Nbr(j)} (\alpha_j + \beta_i)\left(x_j^* - x_j^\circ\right)$$

Under the Pigouvian subsidies, the total subsidy given to the agent j at the optimal action is:

$$B_j^p = \left(\sum_{i \neq j} \beta_i\right)\left(x_j^* - x_j^\circ\right)$$

The ratio of the two budgets for the consumer j at the optimal action is:

$$\begin{aligned}
\frac{B_j^s}{B_j^p} &= \frac{-\sum_{i \in Nbr(j)} (\beta_i + \alpha_j)}{\sum_{i \in N} \beta_i} \\
&< \frac{-\sum_{i \in Nbr(j)} (\alpha_j)}{\sum_{i \in N} \beta_i} \\
&= \frac{-\left|Nbr\,(j)\right| cu_j''\left(x_j^\circ\right)}{u_j'\left(x_j^\circ\right)}
\end{aligned}$$

$$< \frac{|Nbr\,(j)|}{2K} \leq \frac{1}{2}\ \text{, since } c < \frac{-u_j'\left(x_j^{\circ}\right)}{2Ku_j''\left(x_j^{\circ}\right)}$$

Therefore, the total budget in the social mechanism is lower than the total budget in the Pigouvian mechanism. $\sum_{j\in N} B_j^s < \sum_{j\in N} \frac{|Nbr(j)|B_j^p}{2K} < \frac{1}{2}\sum_{j\in N} B_j^s$.

□

4.2 Social Capital

The social mechanism needs a smaller budget for the rewards and thus need lower taxation on the society. Since taxation has overheads and reduces social surplus, lower taxation is better. For simplicity, we assume that the redistribution loss or the overheard introduced due to the taxation and distribution of rewards is a fraction, δ of the total taxation. However, social capital is used in the social mechanism and it is important to find out the total loss of the social capital. We show in the next theorem that the loss incurred in the social capital due to peer-pressure in the equilibrium under social mechanism is smaller than the increase in the redistribution loss in the equilibrium under Pigouvian mechanism over the the equilibrium under social mechanism.

Theorem 4. *The loss in social capital in the equilibrium under the social mechanism is lower than the redistribution loss in the equilibrium under the Pigouvian mechanism for all redistribution factors* $\delta \geq \frac{1}{2K}$.

Proof. The redistribution loss in the equilibrium under the Pigouvian mechanism due to the rewards given to any agent i is

$$\delta u_i'\left(x_i^{\circ}\right)\left(x_i^* - x_i^{\circ}\right).$$

The loss in social capital in the social mechanism is the cost of applying the peer-pressure. There is no loss of social capital due to agents' action because each agent 's action is socially optimal. The total loss in the social capital to the peers of agent i is

$$c\left(p_{\downarrow i}^{\circ}\right) = cu_i'\left(x_i^{\circ}\right)$$

$$= c\left(u_i'\left(x_i^{\circ}\right) - u_i'\left(x_i^*\right)\right), \{\text{Since } u_i'\left(x_i^*\right) = 0\}$$

$$= c\left(\int_{x_i^{\circ}}^{x_i^*} -u_i''\left(x\right)\partial x\right)$$

$$< c\left(\int_{x_i^{\circ}}^{x_i^*} -u_i''\left(x_i^{\circ}\right)\partial x\right),$$

{Since the raw marginal utitlity is strictly convex, therefore $u_i''(x_i^\circ) < u_i''(x)$ for all i and for all $x > x_i^\circ$}

$$= -cu_i''(x_i^\circ)(x_i^* - x_i^\circ)$$

$$< \frac{1}{2K} u_i'(x_i^\circ)(x_i^* - x_i^\circ), \{\text{Since } c < -\frac{1}{2K}\frac{u_i'(x_i^\circ)}{u_i''(x_i^\circ)} \text{ for all } i\}$$

$$\leq \delta u_i'(x_i^\circ)(x_i^* - x_i^\circ).$$

□

Corollary 1. *The total loss in the equilibrium under the social mechanism including the loss in the social capital and the redistribution loss is lower than the the redistribution loss in the equilibrium under the Pigouvian mechanism for distribution factors $\delta \geq \frac{1}{K}$.*

Proof. The redistribution loss in the equilibrium under social mechanism due to the rewards distributed to the peers of any agent i is

$$-\delta K \left(cu_i''(x_i^\circ) + v_i'\left(\mathbf{x}_{-j}^\circ\right)\right)(x_i^* - x_i^\circ)$$

.

Therefore, the total loss in the equilibrium under social mechanism from rewards and social loss to the agents of i is

$$cu_i'(x_i^\circ) - \delta K \left(cu_i''(x_i^\circ) + v_i'\left(\mathbf{x}_{-j}^\circ\right)\right)(x_i^* - x_i^\circ)$$

$$< cu_i'(x_i^\circ) - \delta K cu_i''(x_i^\circ)(x_i^* - x_i^\circ)$$

$$< cu_i'(x_i^\circ) + \frac{1}{2}\delta u_i'(x_i^\circ)(x_i^* - x_i^\circ)$$

$$< \left(\frac{1}{2K} + \frac{\delta}{2}\right) u_i'(x_i^\circ)(x_i^* - x_i^\circ)$$

$$= \left(\frac{1}{2K\delta} + \frac{1}{2}\right) \delta u_i'(x_i^\circ)(x_i^* - x_i^\circ)$$

$$\leq \delta u_i'(x_i^\circ)(x_i^* - x_i^\circ)$$

, since $\delta \geq \frac{1}{K}$.

□

5 Example

We give a simple illustrative example to demonstrate our results. Consider a homogeneous set of agents $N = \{1, ..., 100\}$, each agent has 10 peers in the social network. Each agent consumes electricity priced at 4 per unit. The raw utility function of all agents is $u_i(x_i) = 12x_i^{0.8} - 4x_i$ for all $i \in N$, and the externality function of all agents is $v_i(y) = 0.0001(y)^{1.5}$ for all $i \in N$. The marginal cost of exerting pressure is $c = 1$ per unit. Assume that the redistribution loss is $\delta = 0.1$ per unit of reward. Figure 1 shows how the raw utility of consumption, externality and total utility of each agent changes with the increase in consumption, assuming each agent has the same consumption.

We have a symmetric equilibrium in the absence of peer pressure with $x_i^* = 79.61$ for all $i \in N$. The socially optimal consumption is $x_i^\circ = 31.19$ for all $i \in N$. So the electricity consumption is more than two and a half times the socially optimal level. The ratio of the marginal externality on any agent to the curvature of the raw utility of any other agent at the equilibrium consumption $\mathbf{x}^*$ is $1.33 > c$ and at socially optimum consumption $\mathbf{x}^\circ$ is $0.27 < c$. Therefore the cost of exterting the socially optimal level of pressure on the peers is much higher than the resulting reduction on externality on any agent. The symmetric equilibrium in the model with peer-pressure is better than the model without peer-pressure. The peer-pressure on the agents in the equilibrium is 0.13 which is very low as compared to the optimal peer-pressure 0.83. Therefore, the consumption in the equilibrium is 67.46 which is more than twice the socially optimal consumption. Figure 1 illustrates the situation, highlighting that, even if peer-pressure is possible, it is too costly to apply sufficiently high enough to yield socially optimal consumption.

The total reward budget required to reduce the consumption to the socially optimal level under Pigouvian mechanism will be 3995.40 and the total reward budget under the social mechanism will be 1095.26. Figure 2 shows that the required budget for any target consumption level is lower under the social mechanism than under the Pigouvian mechanism.

The social cost in the equilibrium under the social mechanism will be 82.50. Moreover, the total social cost and the redistribution cost under the social mechanism is 192.03. The total redistribution cost under the Pigouvian mechanism is 399.54 which is one and a half times the cost under the social mechanism. Figure 3 shows that the redistribution and social loss for any target consumption level is lower under the social mechanism than under the Pigouvian mechanism.

References

[1] Mas-Colell A, Whinston M, Green J (1995) *Microeconomic Theory* (Oxford University Press).

[2] Calvó-Armengol A, Jackson M (2010) Peer pressure. *Journal of the European Economic Association* 8:62–89.

[3] Berge C (1963) *Topological Spaces* (Macmillan, New York).

[4] Kuhn HW, Tucker AW (1952) Nonlinear programming. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 481–492.

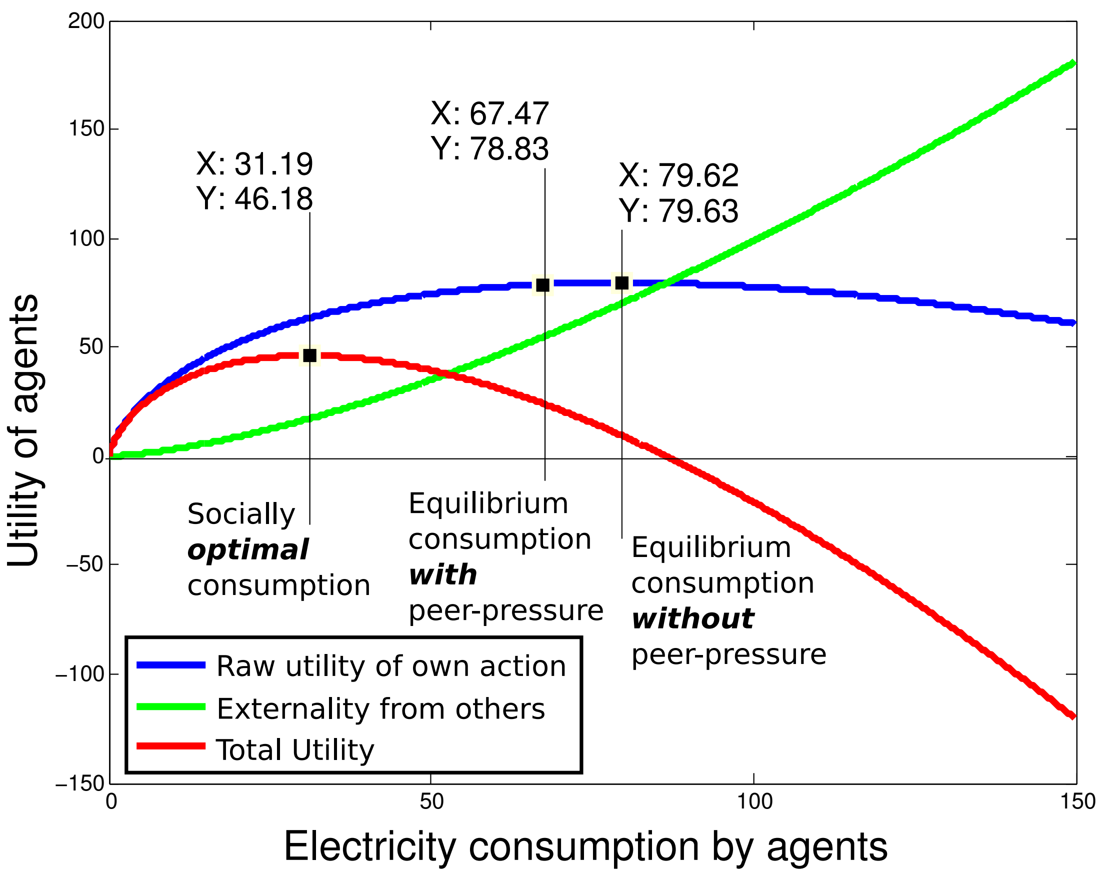

Figure 1: (navy) Raw utility of consumption, maximized at the equilibrium consumption of $X = 79.62$. The peer pressure will lower the equilibrium consumption only slightly to $X = 67.46$ due to high marginal cost of pressure; (green) Externality experienced due to other agents' consumption; (red) Total utility curve is the difference between the navy and green curves, and is maximized at the (much lower) socially optimal consumption level of $X = 31.19$.

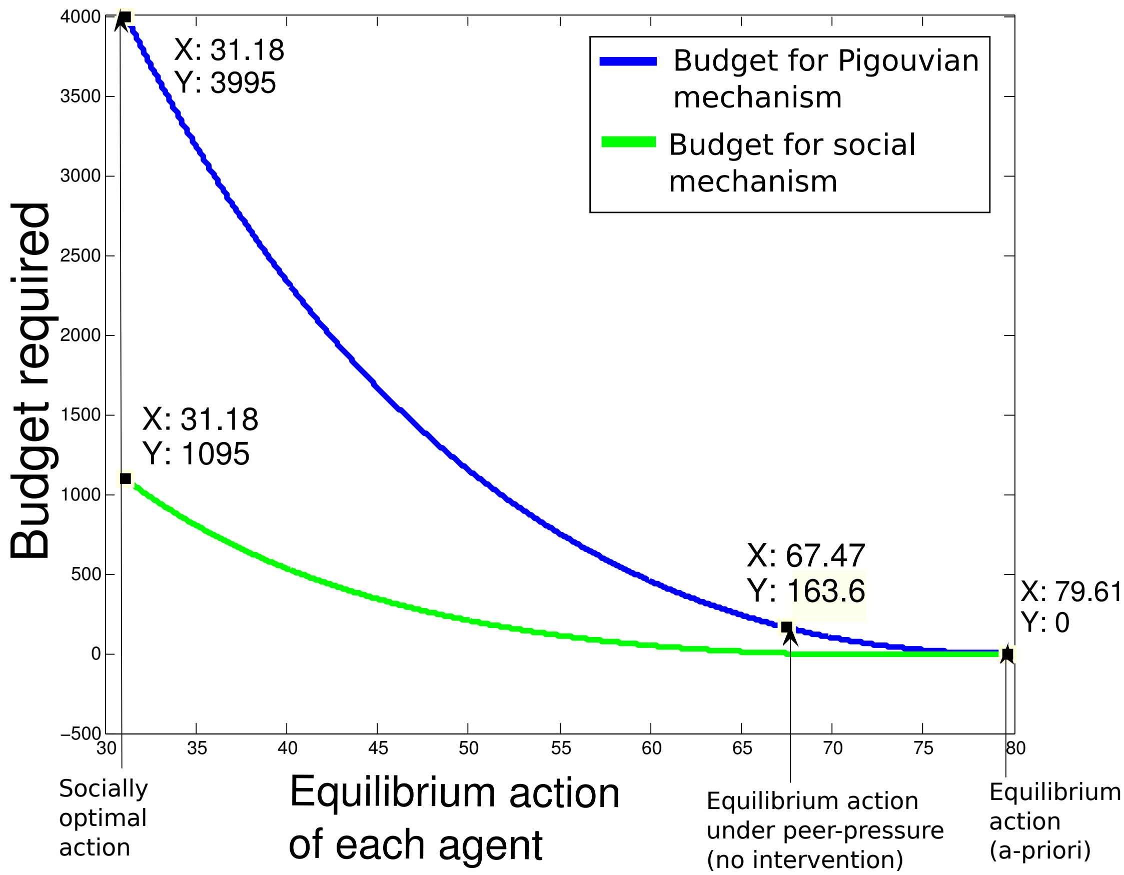

Figure 2: (navy) Budget required for rewards under Pigouvian mechanism for a target consumption level; (green) Budget required for rewards under Social mechanism for a desired consumption level. It increases much slower as the target consumption level is reduced.

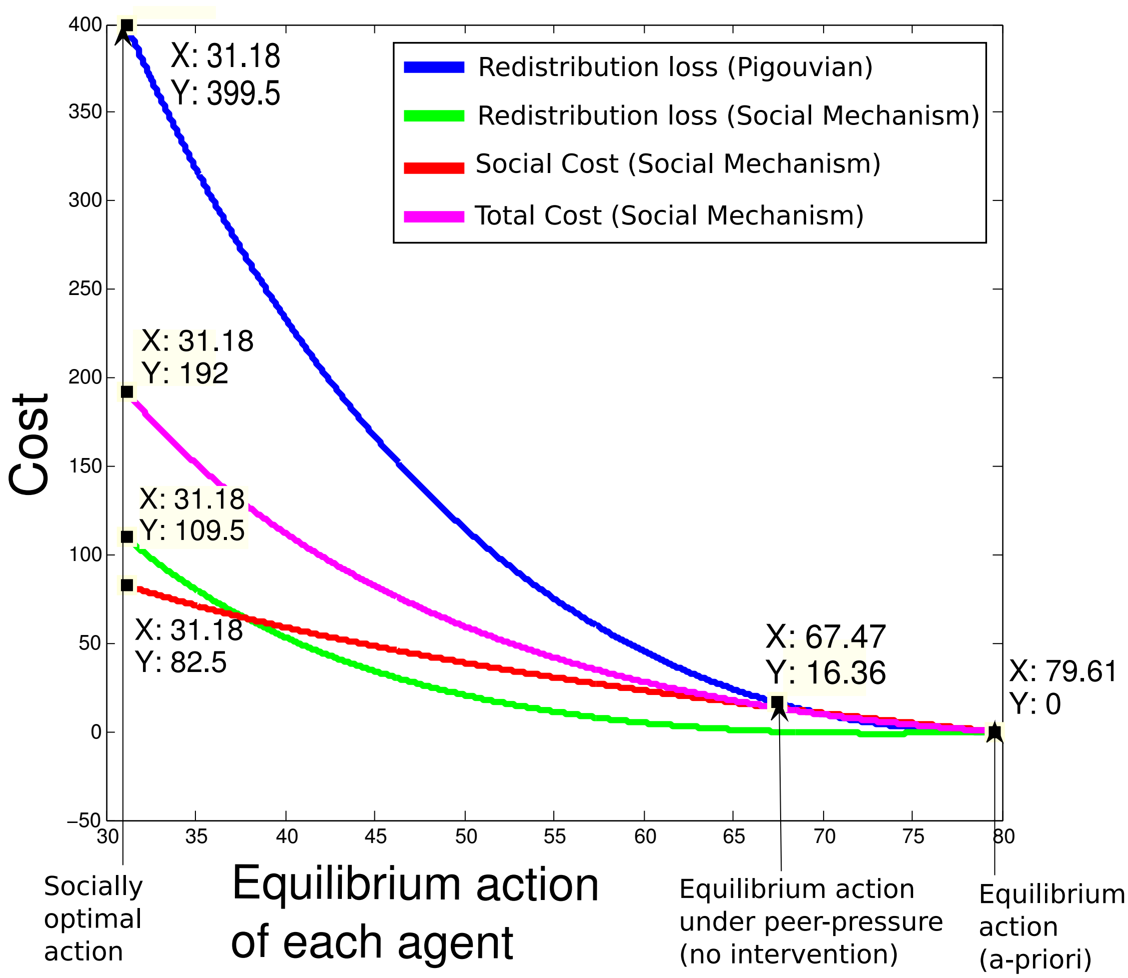

Figure 3: (navy) Redistribution loss under Pigouvian mechanism for a target consumption level; (green) Redistribution loss under Social mechanism for a desired consumption level. It increases much slower as the target consumption level is reduced; (red) Social loss under Social mechanism for a desired consumption level; (magenta) Total (redistribution + social) loss under Social mechanism for a desired consumption level.